Finite Math - J-term 2019

Lecture Notes - 1/28/2019

Homework

- Section 7.4 - 7, 9, 11, 13, 15, 17, 19, 21, 31, 32, 33, 34, 37, 38, 40, 41, 43, 51

Section 7.4 - Permutations and Combinations

There are often situations in which we have to multiply many consecutive numbers together, for example, in examples of the form "from a pool of 8 letters, make words consisting of 5 letters without any repetition." There are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ of these. Let's define a notation that will simplify writing these quantities down:

Definition 1 (Factorial)**.** *For a natural number n,*

$$\begin{aligned} n! &= n(n-1)(n-2) \cdot \cdots \cdot 2 \cdot 1 \\ 0! &= 1 \end{aligned}$$

From this definition, we can see that

$$n! = n \cdot (n-1)! = n(n-1) \cdot (n-2)! = \cdots ,$$

that is, we can explicitly write out as many of the largest numbers as we need, then write the rest as a smaller factorial. For example, we could write

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6!$$

if we wanted to bring special attention to 10 through 7. The following example shows how this is useful.

Example 1. *Find*

(a) $6!$

(b) $\dfrac{10!}{9!}$

(c) $\dfrac{10!}{7!}$

(d) $\dfrac{5!}{0!3!}$

(e) $\dfrac{20!}{3!17!}$

Example 2. *Find*

(a) $7!$

(b) $\dfrac{8!}{4!}$

(c) $\dfrac{8!}{4!(8-4)!}$

0.1. **Permutations.** Suppose we have 5 people to be seated along one side of a long table. There are many possible arrangements of the people, and each of these arrangements is called a permutation.

Definition 2 (Permutation)**.** *A* permutation *of a set of distinct objects is an arrangement of the objects in a specific order without repetition.*

In the set up problem, we have 5 people, and 5 seats to fill. If we fill in the seats from left to right, we can put one of 5 people in the first, one of the remaining 4 in the second, one of 3 in the third, one of 2 in the fourth, and then there is only one person left to fill the fifth seat. Using the multiplication principle, we see that there are

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$$

possible arrangements, or permutations.

Theorem 1 (Permutations of n Objects)**.** *The number of permutations of n distinct objects without repetition, denoted by ${}_nP_n$, is*

Sometimes we don't want to use all of the available options, such as when we're making 5 letter words without repetition out of a pool of 8 letters.

Definition 3 (Permutation of n Objects Taken r at a Time)**.** *A permutation of a set of n distinct objects taken r at a time without repetition is an arrangement of r of the n objects in a specific order.*

Theorem 2 (Number of Permutations of n Objects Taken r at a Time)**.** *The number of permutations of n distinct objects taken r at a time without repetition is given by*

Example 3. *Given the set $\{A, B, C, D\}$, how many permutations are possible for this set of 4 objects taken 2 at a time?*

Solution.

Example 4. *Find the number of permutations of 30 objects taken 4 at a time.*

Solution.

Combinations.

Suppose there is a bag that has 10 jelly beans, each with a different flavor. How many different combinations of 3 flavors can you draw from the bag? Notice that this does not take the order of the flavors into account, but the flavors themselves. In other words, if you draw cherry-grape-apple versus grape-cherry-apple, this difference is not a different combination of flavors.

Definition 4 (Combinations)**.** *A* combination *of a set of n distinct objects taken r at a time without repetition is an r-element subset of the set of n objects. The arrangement of the elements in the subset does not matter.*

Theorem 3 (Number of Combinations of n Objects Taken r at a Time)**.** *The number of combinations of n distinct objects taken r at a time without repetition is given by*

Example 5. *Form a committee of* 12 *people.*

(a) *In how many ways can we choose a chairperson, a vice-chairperson, a secretary, and a treasurer, assuming that one person cannot hold more than one position?*

(b) *In how many ways can we choose a subcommittee of* 4 *people?*

Solution.

Example 6. *Find the number of combinations of* 30 *objects taken* 4 *at a time.*

Example 7. *How many ways can a* 3*-person subcommittee be selected from a committee of* 7 *people? How many ways can a president, vice-president, and secretary be chosen from a committee of* 7 *people?*

Example 8. *Find the number of combinations of* 67 *objects taken* 5 *at a time.*

Example 9. *Suppose we ave a standard* 52*-card deck and we are considering* 5*-card poker hands.*

(a) *How many hands have* 3 *hearts and* 2 *spades?*

(b) *How many hands have all the same suit? (I.e., what is the number of different flushes?)*

(c) *How many possible pairs are there? (The other three cards have a different number from the pair and each other.)*

(d) *How many possible* 3 *of a kinds are there? (The other two cards have a different number from the* 3 *of a kind and from each other.)*

(e) *How many full houses are possible? (A full house consists of a three of a kind and a pair, each from a different number.)*